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ANALYTIC STUDY OF THE HIGH-FREQUENCY IMPEDANCE

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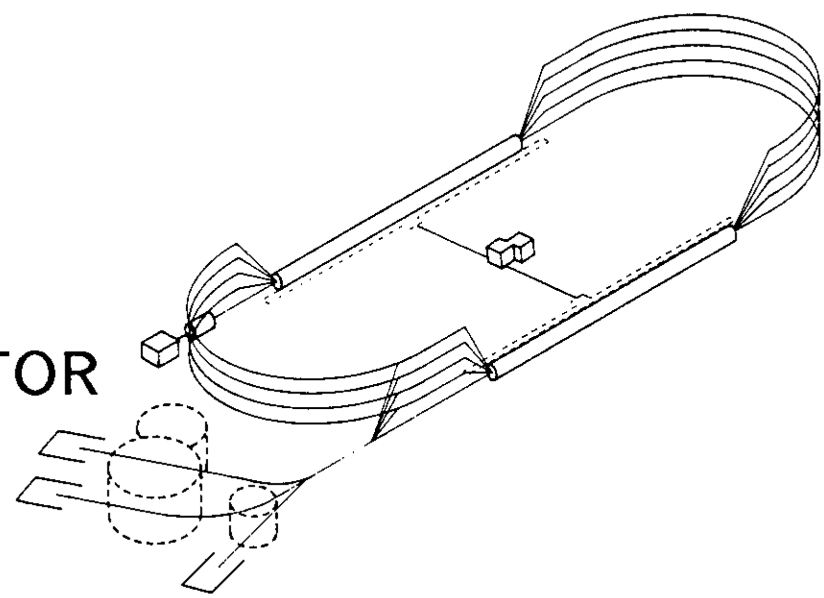

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ABSTRACT

The interaction of short bunches with the beam environment is described using simple models. The impedances of typical elements of an accelerator structure are obtained. The cross-talk between elements, the impedance of a periodic array, and the effect of tapering are discussed. The results allow fast estimate of the high frequency impedance with reasonable accuracy for a cavity of arbitrary shape and give a guideline for impedance optimization.

INTRODUCTION

Stability of a bunch in an accelerator in the relativistic case, where direct interaction between particles is negligible, depends mostly on the interaction with EM wake fields, generated in the system at all variations of the beam pipe cross-section. The wake fields are responsible for such effects as additional heat load, change of energy spread, emittance growth, and collective instabilities.

The wake field can be considered as a response of the system to an external perturbation, which is the beam current. In the linear approximation the Fourier frequency harmonics of the field and of the current are proportional, and for a given frequency the coefficient of proportionality, the impedance, depends only on the geometry and the wall resistivity. The wake potential, related to the impedance by Fourier transformation, describes the bunch-environment interaction in the time domain. The impedance and wake potential are important characteristics of an accelerator.

The impedances can be calculated numerically. The powerful numeric codes such as URMEL, TBCI, MAFIA, and some others are available. Nevertheless, some estimations can be done with reasonable accuracy analytically. This capability is not only useful for rough estimations at the first steps of the accelerator design but also gives a guideline for the optimization of the system. Analytic results are especially useful for very short bunches with a bunch length of a few picoseconds and a frequency content spread up to hundreds of gigahertz, where numeric calculations are limited by an increasing number of mesh points.

In this paper we study the high-frequency behavior of the impedance using simple models based on an approximate formulation of the boundary conditions. In the beginning the impedance and wake potential are defined. After that the impedances are estimated with a simple model for two typical cases: an abrupt change of the beam pipe radius (the

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case referred to later as a step) and the pill-box cavity. In both cases cylindrical symmetry is assumed. These two cases are of interest because most of the impedance generating elements may be approximated by one of them. The elements without cylindrical symmetry such as vacuum ports usually give a smaller contribution to the total impedance. The model is used to discuss the cross-talk between impedance generating elements, the case of the periodic array of cavities, and the effect of tapering. The analytic results are compared whenever possible with results obtained by more rigorous methods and with calculations with TBCI.

THE BASIC FORMULAS

The longitudinal δ-functional wake W_l^δ, by definition [1,2] gives the energy loss ΔE_1 of a particle that follows a point-like bunch with the total charge $Q = eN_b$ at the distance s

$$\Delta E_1 = eQW_l^\delta(s), \quad s > 0. \tag{1}$$

According to this definition, W_l^δ is given by the radiated part of the longitudinal electric wake field $\vec{E}$ coherently excited by the particles of a bunch:

$$\Delta E_1(s) = eQW_l^\delta = -e\int dt\vec{v}\cdot\vec{E}(z = vt - s, t). \tag{2}$$

Note, that the field in the integrand does not include the field of the bunch itself. The radiated part of the field satisfies the homogeneous wave equation and is zero in the straight beam pipe, provided we are not interested in the radiation of the bunch entering or exiting the beam pipe.

Let us define the Fourier transformation as

$$E(t) = \int \frac{d\omega}{2\pi}E(\omega)e^{-i\omega t}.$$

Then

$$W_l^\delta(s) = \int \frac{d\omega}{2\pi}e^{-i\omega s/v}Z_l(\omega) \tag{3}$$

where the impedance $Z_l(\omega)$ is

$$Z_l(\omega) = -\frac{1}{Q}\int dzE_z^\omega(z,r)e^{-i\omega z/v}. \tag{4}$$

From the casuality principle it follows that $W_l^\delta(s) = 0$ for $s < 0$. Hence, $Z_l(\omega)$ may have poles in the complex plane of ω only for Im $\omega < 0$. Narrow-band impedance can be expressed as a sum of Breit-Wigner terms, each of which describes a mode with the frequency ω_λ, the decrement $\gamma_\lambda = \omega_\lambda/Q_\lambda$ and the loss-factor χ_λ

$$Z_l(\omega) = i\sum_\lambda \chi_\lambda(\frac{1}{\omega - \omega_\lambda + i\gamma_\lambda} + \frac{1}{\omega + \omega_\lambda + i\gamma_\lambda}). \tag{5}$$

According to Eq. (3) the wake in this case is given as

$$W_l^\delta(s) = 2\sum_\lambda \chi_\lambda \cos(\frac{\omega_\lambda s}{v}) e^{-\gamma_\lambda s/v}. \tag{6}$$

In Eq. (6) we have assumed that $s > 0$. For $s = 0$

$$W_l^\delta(s) = \sum_\lambda \chi_\lambda.$$

The energy loss into a single mode for a particle in a $\delta-$ function bunch with N_B particles is

$$\Delta E_\lambda = e^2 N_B \chi_\lambda.$$

The loss factor χ_λ can be found if the eigenfunction of the mode E_z^λ are known from the relation

$$\chi_\lambda = \frac{|V_\lambda|^2}{4U_\lambda}, \tag{7}$$

where

$$V_\lambda = \int dz e^{-ikz} E_z^\lambda(r,z)$$

and U_λ is the energy, stored in a mode. The factor χ_λ is proportional to the ratio of the shunt impedance r/Q

$$\chi_\lambda = \frac{\omega}{4}\frac{r_\lambda}{Q_\lambda} \tag{8}$$

and, practically, can be calculated by the numeric code URMEL. Note that URMEL defines r in such way, that factor 1/2 has to be used in Eq. (8) instead of the factor 1/4.

The average energy loss k_l of a particle in a bunch with longitudinal density $\rho(z)$, $\int \rho dz = 1$ is

$$k_l = <\frac{\Delta E_1}{eQ}> = \int dz_1 dz_2 \rho(z_1)\rho(z_2) W_l^\delta(z_1 - z_2). \tag{9}$$

For a Gaussian bunch with rms $< z^2 >= \sigma^2$ the density

$$\rho(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/2\sigma^2}$$

has Fourier harmonics

$$\rho(k) = e^{-k^2\sigma^2/2} \quad k = \omega/c$$

and k_l can be written as

$$k_l = \int \frac{d\omega}{2\pi} Z_l^\delta(\omega)|\rho(k)|^2. \tag{10}$$

For a train of bunches the wake fields excited by all bunches preceding the test particle have to be taken into account. The interference of these fields changes the loss by the factor $F(ks_B/2Q, ks_B)$, where

$$F(x,y) = \frac{\sinh\ x}{\cosh\ x - \cos\ y} \tag{11}$$

and s_B is the bunch spacing. If the Q-factor is large, $Q \gg 1$, and there is resonance excitation $ks_B = 2\pi n$, the loss can be substantially enhanced by $F \sim 4Q/ks_b$. This situation can occur for beam current monitors which use a harmonic of the fundamental frequency $k = nk_f$, $k_f = 2\pi/s_b$.

The transverse δ-functional wake gives the average transverse kick caused by the wake field of a bunch:[1]

$$W_{\perp}^{\delta}(s,r) = \frac{1}{Q}\int dz(\vec{E}_{\perp} + (\vec{v}\times\vec{B})_{\perp})_{vt=s+z}. \tag{12}$$

The transverse impedance $Z_{\perp}$ is Fourier related to $W_{\perp}$

$$W_{\perp}^{\delta} = i\int \frac{d\omega}{2\pi} Z_{\perp} e^{-i\omega s/v}$$

$$iZ_{\perp}(r,\omega) = \frac{1}{Q}\int dz(\vec{E}_{\perp} + \frac{1}{c}(\vec{v}\times\vec{B})_{\perp})e^{-i\omega z/v}. \tag{13}$$

In modal analysis $W_{\perp}$ can be expressed similarly to Eq. (6)

$$W_{\perp}^{\delta} = 2\sum_{\lambda}\chi_{\perp}^{\lambda}\sin(\omega_{\lambda}s/v). \tag{14}$$

Notice, that $W_{\perp}^{\delta} = 0$ at $s = 0$.
The transverse loss $\chi_{\perp}^{\lambda}$ is given by the eigenfunctions of the modes similarly to Eq. (7):

$$\chi_{\perp}^{\lambda} = \frac{c}{\omega_{\lambda}}[\nabla_{\perp}\frac{V_{\lambda}^{*}(r')\, V_{\lambda}(r)}{U_{\lambda}}]_{r=r'}. \tag{15}$$

The transverse loss is defined as

$$k_{\perp} = < \frac{W_r}{a} > = i\int \frac{d\omega}{2\pi a}|\rho(k)|^2 Z_{\perp}(r=a,\omega) \tag{16}$$

There are several general statements on the impedances. First, the wakefield $W(s)$ has to be real. Hence, $Z^*(\omega) = Z(-\omega^*)$. In particular, the imaginary part $\mathrm{Im}Z(0) = 0$. Secondly, as was mentioned before, from the casuality principle it follows that $Z_l(\omega)$ may have poles in the complex plane of ω only for Im $\omega < 0$. The Cauchy theorem gives the dispersion relations between real Z' and imaginary Z'' parts of the impedance:

$$Z'(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\omega' Z''(\omega')}{\omega'-\omega}$$

$$Z''(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\omega' Z'(\omega')}{\omega'-\omega}.$$

Here the integral is taken as a principal value. Hence, the real part of the impedance defines the imaginary part.

The Panofsky-Wenzel theorem[3] relates the longitudinal and transverse wakes for a given mode:

$$\frac{\partial}{\partial s} W^{\delta}_{\perp,\lambda} = \nabla_{\perp} W^{\delta}_{l,\lambda}. \tag{17}$$

It can be easily deduced from the definitions Eq. (4) and Eq. (13) using the Maxwell equation $ik\vec{B} = \nabla \times \vec{E}$. Eq. (17) gives after the Fourier transform1

$$Z_{\perp} = \frac{v}{\omega} \nabla_{\perp} Z_l. \tag{18}$$

Another theorem, given by T. Weiland[4], shows that the radial dependence of the impedances in the ultrarelativistic case is known. To prove the theorem, it is enough to notice, that the radiated part of the EM field satisfies a homogeneous wave equation

$$\Delta \vec{E}^{\omega}_z + k^2 \vec{E}^{\omega}_z = 0$$

and that for ultrarelativistic particles the impedances are given by the synchronous components of the field:

$$E^{\omega}_z(k, r, \phi) = \int dz \vec{E}^{\omega}_z(z, r, \phi) e^{-ikz}.$$

The azimuthal components

$$E^{\omega}_z(k, r, \phi) = \sum E^{\omega,m}_z(k, r)\ e^{im\phi}$$

satisfy in the ultrarelativistic case the equations

$$(\Delta_r - \frac{m^2}{r^2}) E^{\omega,m}_z(k, r) = 0$$

where

$$\Delta_r = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}.$$

The field components have to be finite at $r = 0$. This gives

$$Z_l(k, r) \propto E^{\omega,m}_z(k, r) = w_m(k) r^m$$

The components defining the transverse impedance are

$$Z_r(k, r) \propto [\vec{E}(k, r) + \frac{1}{c}\vec{v} \times \vec{B}(k, r)]^{\omega,m}_r = -\frac{im}{k} w_m(k) r^{m-1}$$

$$Z_{\phi}(k, r) = i Z_r(k, r).$$

---

1 J.Bisognano noticed that a $\delta(\omega)$ functional term may be added to the right side of Eq. (18). The transverse wake has in this case a time-independent part.

All components are given by the same functions $w_m(k)$. Therefore, the transverse impedance can be deduced if the longitudinal impedance is known in accordance with the Panofsky-Wenzel theorem. The main contribution to the longitudinal impedance Z_l gives the monopole mode $m = 0$ and does not depend on r. Therefore, the integration in the right-hand side of Eq. (4) can be performed with arbitrary offset along the beam pipe axes. The convenient choice of the offset is the beam pipe radius. In this case the integration in Eq. (4) may be performed only in the cavities, because of the tangential component of the field $E_z = 0$ on the perfectly conductive walls.

Recently R. Gluckstern and B. Zotter have given a theorem that the impedance of a rotationally symmetric cavity is independent of the direction in which the beam travels[20].

The modal analysis based on the Eq. (7) and Eq. (15) is useful for narrow-band impedance for frequencies below or close to the beam pipe cutoff. In the high frequency extreme a large number of modes become important. It is adequate in these cases to use diffractional models. In the next section the model based on the approximate formulation of the boundary conditions is described in the high-frequncy limit $1 \ll ka \ll \gamma$*

For convenience we give the relation between different units:

$$1\ V/pC = 1.11\ cm^{-1}$$

SIMPLE MODELS FOR HIGH-FREQUNCY IMPEDANCE

A Bunch in Free Space and in a Beam Pipe

The field of a free particle with the charge e moving with velocity v in z direction is the solution of the wave equation

$$\Delta \vec{E}_\omega + k^2 \vec{E}_\omega = 4\pi \nabla \rho_\omega - \frac{4\pi i \omega}{c^2} \vec{j}_\omega$$

where $k = \omega/c$ and the Fourier components of the current $\vec{j}_\omega = \hat{z} j_\omega$ and density ρ_ω are

$$j_\omega = e\rho(\vec{r}_\perp)e^{ikz}, \quad \rho_\omega = j_\omega/v, \quad \int \rho(\vec{r}_\perp)d^2r_\perp = 1.$$

For a point-like particle

$$E_{\omega,r} = \frac{2e\tau}{v}K_1(\tau r)e^{ikz/\beta}, \quad E_{\omega,z} = \frac{-2ie\tau^2}{\omega}K_0(\tau r)e^{ikz/\beta}$$

where $\beta = v/c$ and $\tau = k/\beta\gamma$. This field propagates synchronously with the particle.

For an ultrarelativistic particle $E_{\omega,z} = 0$. $E_{\omega,r}$ is exponentialy small for $r > \gamma/k$ and

$$E_{\omega,r} = \frac{2e}{vr}e^{ikz} \quad \text{for} \quad r < \gamma/k. \tag{19}$$

---

* It is easy to see that for $ka > \gamma$ impedances are exponentially small. Perfectly conductive walls are assumed everywhere.

The total energy flow across the plane $z = \text{const}$ is given by Pointing vector

$$W = \int_{-\infty}^{\infty} dt \int_0^{\infty} 2\pi r dr \frac{c}{4\pi}(\vec{E} \times \vec{H})_z = \frac{c}{4\pi}\int_0^{\infty} 2\pi r dr \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}|E_{\omega,r}|^2. \tag{20}$$

With $E_{\omega,r}$ given by Eq. (18) the integral diverges at small r:

$$W \sim \frac{e^2\gamma}{r_{\min}}.$$

If $r_{\min}$ is the classical electron radius $r_0 = e^2/mc^2$ the energy of the field synchronous component is of order of energy of the particle $W \sim E$ and actually depends on the definitions of the electron mass and of the charge of a free particle.

For a free rigid bunch with rms bunch length σ and the total charge $Q = e^2 N_B$ the integration over frequencies in Eq. (20) is limited to $k < 1/\sigma$. The main contribution is given by the area $|k| < 1/\sigma$, $0 < r < |\frac{\gamma}{k}|$ where the fields of particles are added coherently. That gives the energy flow per particle:

$$W = \frac{N_B e^2}{2\pi}\left(\int_0^{\gamma\sigma} \frac{dr}{r}\frac{1}{\sigma} + \int_{\gamma\sigma}^{\infty} \frac{dr}{r}\frac{\gamma}{r}\right) = \frac{N_B e^2}{2\pi\sigma}\left(\ln\frac{\gamma\sigma}{r_{\min}} + 1\right).$$

For a particle in a pipe with the radius a

$$E_{\omega,r} = \frac{2e\tau}{v}\left[K_1(\tau r) + I_1(\tau r)\frac{K_0(\tau a)}{I_0(\tau a)}\right]e^{ikz},$$

$$E_{\omega,z} = \frac{-2ie\tau^2}{\omega}\left[K_0(\tau r) - I_0(\tau r)\frac{K_0(\tau a)}{I_0(\tau a)}\right]e^{ikz}.$$

The first terms here are the same as for a free particle. The second terms give the field of the image current in the beam pipe wall. They are small for $\gamma \to \infty$ and the field is the same as for a free particle. For the radius $a \ll \gamma\sigma$ the field energy of a bunch per particle is

$$W = \frac{N_B e^2}{2\pi\sigma}\ln\left(\frac{a}{r_{\min}}\right).$$

If the pipe radius is changed from $r = a$ to $r = b > a$ the energy of the synchronous component of the field increases. It corresponds to the energy loss per particle[8]

$$\Delta E = \frac{N_B e^2}{2\pi\sigma}\ln\frac{b}{a}. \tag{21}$$

Impedances of a Cavity and a Step

Let us consider now a particle exiting out of the beam pipe which is attached to a perfectly conductive plane at $z = 0$. The total field at $z > 0$ is the superposition of the synchronous component and the radiated field:

$$E_{\omega,r} = \frac{2e\tau}{v}K_1(\tau r)e^{ikz} + \int_{-\infty}^{\infty} q dq A_k(q) J_1(qr) e^{iz\sqrt{k^2-q^2}}. \tag{22}$$

Here $A_k(-q) = A_k(q)$ and the path of the integration goes in q-plane above the cut for $q > k$ and below the cut for $q < -k$.

The field has to be matched to the field in the pipe at $z = 0$. It would give an integral equation for $A_k(q)$. Instead, we neglect the reflected wave in the pipe. That can be done for E_r component only, because the z-component of the incident wave in the pipe is zero in this approximation. That is why we choose the r component of the field for matching. In this approximation the field in the pipe is given by Eq. (19) and the matching gives the condition

$$\int q dq A_k(q) J_1(qr) = -\frac{2e}{cr}\theta(r-a) \tag{23}$$

where $\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$. Here we used the approximation $K_1(\tau r) \approx 1/(\tau r)$ for the significant r, $r \ll \gamma/k$. Equation (22) means that the radiated field is defined by the image charge on the surface $r > a, z = 0$. To describe a cavity with a radius $b > a$ the only requirement is to change the right hand side of Eq. (23) which is not zero only for $a < r < b$. The exact boundary condition at $r = b$ for a large cavity is not crucial, so that Eq. (22) is still a good approximation. We consider hereafter a cavity as a more general case. Orthogonality of the Bessel functions gives $A_k(q)$ $(q > 0)$:

$$A_k(q) = -\frac{2e}{c}\int_a^b dr J_1(qr) = -\frac{2e}{cq}[J_0(qa) - J_0(qb)]. \tag{24}$$

The z-component of the diffracted wave can be obtained now from Eqs. (22), (24) and $div\vec{E}_\omega = 0$:

$$E_{\omega,z} = -\frac{2iek}{c}e^{ikz}F(ka, kr, kz) \tag{25}$$

where

$$F(ka, kr, kz) = \int_0^\infty \frac{xdx}{\sqrt{1-x^2}}[J_0(kax) - J_0(kbx)]J_0(krx)e^{ikz(\sqrt{1-x^2}-1)}. \tag{26}$$

For $ka \gg 1$ substantial contribution is given by small $x \ll 1$. To find the impedance we integrate $E_{\omega z}(r = a)$ given by Eq. (25) over $0 < z < g$:

$$Z_l(k) = \frac{Z_0}{\pi}\int_0^\infty \frac{dy}{y}J_0(y)[J_0(y) - J_0(\frac{b}{a}y)](1 - e^{-i\lambda^2 y^2}) \tag{27}$$

where $\lambda^2 = g/2ka^2$ and $Z_0 = 4\pi/c = 377 Ohm$. It can be estimated easily:

$$Z_l(k) = Z_0 \begin{cases} \frac{(1+i)}{\sqrt{2\pi}}\lambda/\pi & \text{for } \lambda < 1 \\ \frac{1}{\pi}\ln\lambda & \text{for } b/a > \lambda > 1 \\ \frac{1}{2\pi}\ln b/a + \frac{1}{\pi} & \text{for } \lambda > b/a \end{cases} . \tag{28}$$

For $g \ll ka^2$ (we call this case "a cavity")

$$Z_l(k) = (1+i)\frac{Z_0}{2\pi a}\sqrt{\frac{g}{\pi k}}. \tag{29}$$

For $kb^2 \gg g \gg ka^2$

$$Z_l(k) = \frac{Z_0}{2\pi} \ln(\frac{g}{ka^2}). \tag{30}$$

If g is large, the cavity radius b becomes important. For $g \gg kb^2$

$$Z_l(k) = \frac{Z_0}{\pi} \ln(\frac{b}{a}). \tag{31}$$

We call this case "a step".

The real part of the impedance gives the energy loss Eq. (10). If $g\sigma/a^2 \ll 1$ the loss is given mostly by the high-frequency part of the impedance

$$k_l = \frac{\Gamma(1/4)}{\pi a}\sqrt{\frac{g}{\pi\sigma}}, \quad \frac{\Gamma(1/4)}{\pi} = 1.154. \tag{32}$$

The low frequencies $k < g/a^2$ give small additional contribution $2g/\pi a^2$.

If $g\sigma/a^2 \gg 1$ the loss given by high-frequency part of the impedance is exponentialy small and the main contribution is given by the the low frequencies $k < g/a^2$

$$k_l = \frac{1}{\sigma\sqrt{\pi}} \ln(\frac{g\sigma}{a^2}). \tag{33}$$

For a step g has to be replaced by kb^2. It gives

$$k_l = \frac{2}{\sigma\sqrt{\pi}} \ln(\frac{b}{a}) \tag{34}$$

which is different from Eq. (21) only by a numeric factor.

The difference between the result for a cavity and the result for a step corresponds to two different regimes of diffraction. For small $z \ll kr^2$ the transverse size of the diffracted wave increases as $r \sim \sqrt{\frac{2z}{k}}$ and the angle of diffraction $\theta \sim \frac{1}{\sqrt{kz}}$, i.e., Fresnel diffraction. For large z the Fraunhofer diffraction takes place for which $r \sim z/ka$, $\theta \sim 1/ka$. For a step all waves with $k < 1/\sigma$ are diffracted in the Fraunhofer regime giving Eq. (31). For a cavity Fraunhofer diffraction takes place only for long waves with $k < g/a^2 < 1/\sigma$ giving Eq. (29).

Equation (29) was obtained by Lawson[9] for a single particle and by G. Dome[10] who conjectured that the radial field in the cavity at $r = a$ is the same as that in a pipe. This assumption was clarified in our paper[11] where rigorous matching was carried out and the correction to Eq. (29) was estimated. Equation (29) was confirmed also in numeric calculations[12] and in diffractional models.[5,6,7]

Impedance of a step was studied by numerical solution of the truncated system of equations obtained by the matching method.[13] It is shown in this paper that impedance for high frequencies is approximately independent of frequency. It is substantially different from zero only for a particle entering a wider pipe, whereas the impedance is almost zero for a particle entering a narrower pipe. For small frequencies, however, the impedance in the last case is negative giving some total energy gain. This acceleration reflects attraction of a particle by the image charge in the wall of the cavity. Impedances of the two steps are different because the waves generated in two cases propagate in opposite directions. Although both waves take out some energy, in the first case the energy of the field is taken out of the energy of a particle, whereas in the second case the wave carries out the excessive energy of the synchronous component of the field.

We checked[14] Eq. (29) with the code TBCI with parameters chosen to be close to CEBAF parameters for FPC (a=3.5 cm, g=2.5 cm, cavity radius b=5.65 cm), HOM (a=3.75 cm, g=3.75 cm, b=5.5 cm) and (3) gate valve (a=1.75 cm, g= 2 cm, b= 3.5 cm). In all three cases the dependence of the average loss vs rms bunch size in the range σ = 0.75 mm − 1.5 mm corresponds to Eq. (29), and numeric agreement is within 10% accuracy.

The transverse loss can be estimated using Eqs. (16), (18), (4) and (25). Note that

$$\frac{\partial F(ka,kr,kz)}{\partial r}|_{r=a} = \frac{1}{2}\frac{\partial F(ka,ka,z)}{\partial a}.$$

Equation (16) takes the form

$$k_{\perp} = \frac{1}{a}\frac{\partial}{\partial a}\int_0^{\infty}\frac{dk}{\pi}e^{-k^2\sigma^2}\int_0^g dzRe\ F(ka,ka,z).$$

For the estimate we truncate the integral at $k < 1/\sigma$. Because $k_{\perp}(\sigma = 0) = 0$ the integral can be replaced by the integral over $k > 1/\sigma$. Using Eq. (27) for F we obtain

$$k_{\perp} = \int_{1/\sigma}^{\infty}\frac{dk}{\pi a}[\frac{1}{ka^2}\sqrt{\frac{\min(g,ka^2)}{\pi k}} - \theta(g-ka^2)(\frac{2}{ka}-\frac{2a}{g})].$$

For a cavity, $g \ll a^2/\sigma$,

$$k_{\perp} \propto \frac{1}{a^3}\sqrt{\pi g\sigma}.$$

Numerically[14] the transverse loss in the form

$$k_{\perp} = \frac{1}{a^3}\sqrt{\pi g\sigma} \tag{35}$$

agreed with TBCI with very good (15%) accuracy. If $g \gg a^2/\sigma$ the main contribution is given by

$$k_{\perp} = \frac{2}{\pi a^2}[1-\ln(\frac{g\sigma}{a^2})]. \tag{36}$$

The simulations with TBCI[14] confirmed that $k_{\perp}$ decreases logarithmically for long bunches, whereas it increases with σ according to Eq. (35) for short bunches.

Equations (32-36) are very useful for the fast estimate of the impedances because a major contribution is given by the structures which can be approximated as a pill-box cavity or steps. Numeric analysis with TBCI confirmed that the transition from a regime of a cavity to the regime of a step depends on the parameter[14,15]

$$p = \frac{2g\sigma}{(b-a)^2}. \tag{37}$$

In particular, for $p \ll 1$, impedances do not depend on the cavity radius. Energy loss decreases with σ as $1/\sqrt{\sigma}$ whereas the transverse impedance increases as $\sqrt{\sigma}$. For $p \gg 1$, qualitative dependence on the parameters is in agreement with the formulas for the step regime. Unfortunately, the available version of TBCI ($3. \times 10^5$ mesh points) does not allow us to perform more comprehensive verification of the Eqs. (32-36).

Another problem, very similar to what we considered, is the impedance of two semi-infinite pipes: $r = a$ for $z < 0$, and $r = b$ for $z > 0$. This problem is probably the only known problem which was solved exactly.[19] It is easy to see that this problem is equivalent to a problem of a single washer $a < r < b$ at $z = 0$, which gives the boundary condition Eq. (23). Therefore, our results for a washer also have to be valid for this problem. This is indeed the case. [19]

Cross-Talk Between Cavities

The wake field generated in one element of an accelerator propagates into the adjacent elements downstream of the system and interfere with the local wake fields. Even if the impedances of all elements, considered as independent, are known it is not clear in general how the interference affects the impedance of the whole system. We consider two examples: two adjacent cavities and, in the next section, the periodic structure as the extreme case.

Let us consider two adjacent cavities with the lengths (g_1 and g_2) and radii (b and d) correspondingly, so that the radius of the system is

$$r = a,\ z < 0;\quad r = b,\ 0 < z < g_1;\quad r = d,\ g_1 < z < g_1 + g_2;\quad r = a,\ z > g_1 + g_2.$$

For simplicity we consider $d \gg b \gg a$ and take $d \to \infty$. The radial component of the wave diffracted at $z = 0$ is the same as for a single cavity:

$$E_{r\omega} = -\frac{2ek}{c} e^{ikz} F_r(z, r) \tag{38}$$

where

$$F_r = \int_0^\infty dx [J_0(kax) - J_0(kbx)] J_1(krx) e^{ikz(\sqrt{1-x^2}-1)}\ , \quad 0 < z < g_1. \tag{39}$$

The field at $z > g_1$ has the same form as Eq. (22), however, the matching condition at $z = g_1$ has to take into account the wave Eq. (38). That gives the diffracted wave at $z > g_1$ in the form of Eq. (38) with

$$F_r = \int_0^\infty dx J_1(krx) e^{ikz(\sqrt{1-x^2}-1)} [J_0(kax) - J_0(kbx) + J_0(kbx) e^{-ikg_1(\sqrt{1-x^2}-1)}]. \tag{40}$$

The z-component of the field for $g_1 < z < g_2$ is given by Eq. (25) where

$$F = \int_0^\infty \frac{xdx}{\sqrt{1-x^2}} J_0(krx)e^{ikz(\sqrt{1-x^2}-1)}[J_0(kax) - J_0(kbx) + J_0(kbx)e^{-ikg_1(\sqrt{1-x^2}-1)}] \tag{41}$$

and by Eq. (25) and Eq. (26) for $0 < z < g_1$.

Let $Z(g,a,b)$ be the impedance Eq. (28) of a single cavity with the length g, the cavity radius b and the beam pipe radius a. The longitudinal impedance takes the form

$$Z_l(\omega) = Z(g_1 + g_2, a, b) + Z(g_2, b, d) + \delta Z \tag{42}$$

where the last term describes the cross-talk between cavities:

$$\delta Z(\omega) = Z(g_2, a, b) - \frac{2i}{cka^2} \int_0^{g_2} dz \int_0^\infty xdx[J_0(x) - J_0(\frac{b}{a}x)]^2 e^{-i\frac{zx^2}{2ka^2}}. \tag{43}$$

The estimate of Eq. (43) shows that for a "cavity regime" δZ gives a correction which cancels the second term in Eq. (42). Hence, the total impedance in this case is the impedance of the cavity $Z(g_1 + g_2, a, b)$. This confirms the statement that the impedance does not depend on the shape of the cavity[21] and is true for the parameters of the cavity that correspond to the "cavity regime".

Periodic Array

For an array of cavities where the number of cavities is large, the interference can be enchanced and can drastically change the impedance. Let us consider a periodic array of diaphragms with the iris radius a separated by the distance L. The field at the iris is the sum of the field of a particle Eq. (19) and the diffracted wave $f(k,r)$:

$$E_{\omega,r} = \frac{2e}{cr}e^{ikz} + \frac{2e}{c}f(k,r)e^{ikz}. \tag{44}$$

Matching the field Eq. (22) between irises and the field Eq. (44) defines the coefficients $A(q)$ and the diffracted field between diaphragms:

$$E_r^{rad}(z,r) = -\int_0^\infty qdqJ_1(qr)e^{iz\sqrt{k^2-q^2}}[\frac{2e}{cq}J_0(qa) - \frac{2e}{c}\int_0^a rdrf(k,r)J_1(qr)]. \tag{45}$$

At $z = L$, $r < a$ it follows from periodicity, that

$$E_r^{rad}(L,r) = \frac{2e}{c}f(k,r)e^{ikL}. \tag{46}$$

This gives the integral equation for $f(k,r)$:

$$f(k,r) = -\int_0^\infty dqJ_1(qr)J_0(qa)e^{-iLq^2/2k} + \int_0^\infty qdqJ_1(qr)e^{-iLq^2/2k}\int_0^a r'dr'f(k,r')J_1(qr'). \tag{47}$$

Note, that $f(0) = 0$.

Equation (47) can be simplified using[16]

$$\int_0^\infty qdqJ_\lambda(qr)J_\lambda(qr')e^{-i\nu q^2/2} = \frac{-i}{\nu}J_\lambda(rr'/\nu)e^{\frac{i}{2\nu}(r^2+r'^2)-\frac{i}{2}\pi\lambda} \quad (48)$$

to the form

$$\sqrt{r}f(k,r) = -\sqrt{r}\int_0^\infty dqJ_1(qr)J_0(qa)e^{-iLq^2/2k} + \int_0^a dr'\Psi(r,r')\sqrt{r'}f(k,r'). \quad (49)$$

Here

$$\Psi(r,r') = -i\frac{k}{L}\sqrt{rr'}J_0(\frac{k}{L}rr')e^{i\frac{k}{2L}(r^2+r'^2)}.$$

The function Ψ is a sharp function of $(r-r')$

$$\Psi(r,r') \simeq \sqrt{\frac{k}{2\pi L}}e^{-i\frac{\pi}{4}+i\frac{k}{2L}(r-r')^2}. \quad (50)$$

For $ka^2/L \gg 1$ $\Psi(r,r')$ represents the δ- function

$$\lim(k\to\infty)\Psi(r,r') = \delta(r-r')$$

and Eq. (49) gives in this limit $af(a) = -1$. Therefore, the radiated field $E_{\omega,r}$ at the iris increases from $f = 0$ at $r = 0$ to $f \approx= -1/a$ at $r = a$ and decreases as $-1/r$ for $r > a$. For a single cavity $E_{\omega,r}$ has the discontinuity at $r = a$ changing from zero to $-1/a$. Here, we introduce the function

$$\frac{\partial}{\partial r}rf(k,r) = -\Lambda\sqrt{\frac{r}{a}}F(a-r), \quad (51)$$

where

$$\Lambda = 1 + af(a). \quad (52)$$

It satisfies the integral equation

$$F(a-r) = \Psi(a,r) + \int_0^a dr'F(a-r')\Psi(r,r') \quad (53)$$

with the same kernel Ψ as in Eq. (49).

We solved Eq. (49) and Eq. (53) numerically for different frequencies k. The typical behavior of the function $rf(k,r)$ is depicted in Fig. 1. The parameter $\Lambda(k)$ has been found from these calculations and its dependence on $ka^2/2L$ is shown in Fig. 2. It scales as

$$\Lambda \approx \frac{4\pi L}{ka^2}. \quad (54)$$

Insignificant oscillations of $\Lambda(k)$ have been found also in the reference[17]. The function $F(a-r)$ oscillates rapidly. (See Fig. 3)

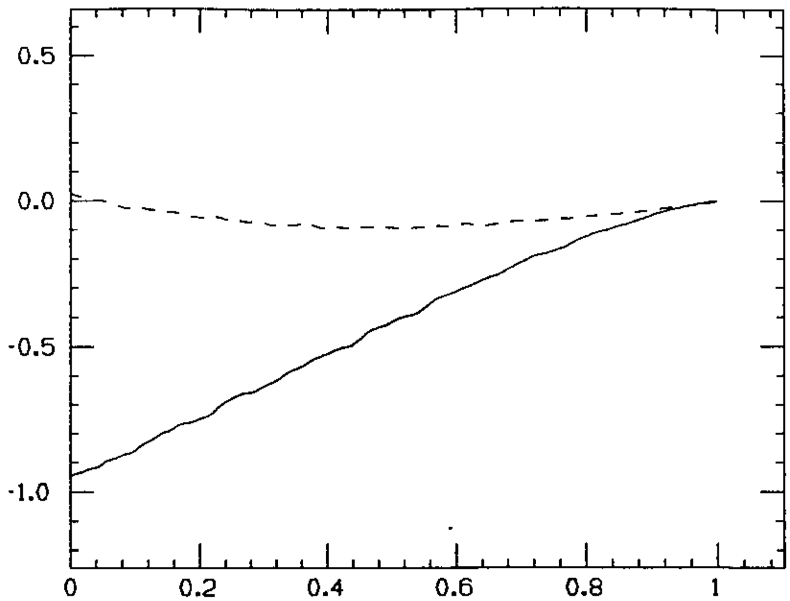

Figure 1 Re(rf) (sld) and Im(rf) (dash) vs. (a-r)/a. $ka^2/2L = 125$

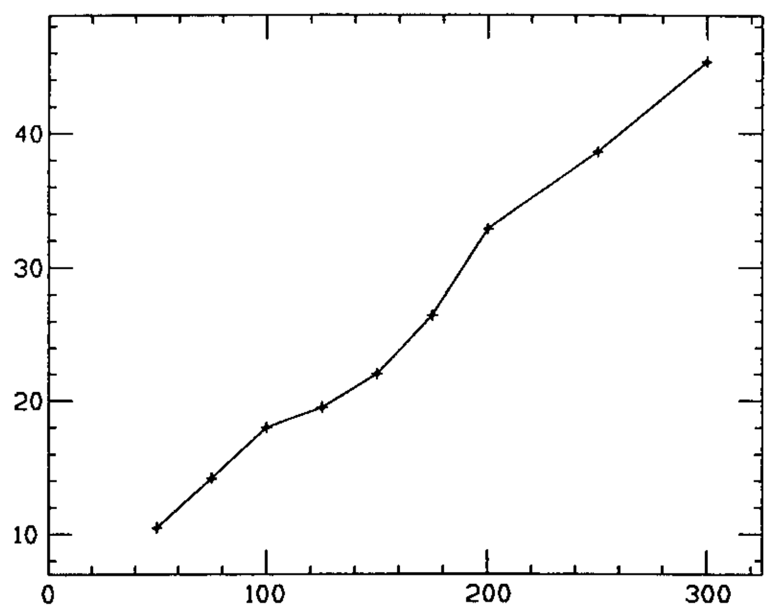

Figure 2 $1/\Lambda$ vs. $ka^2/2L$

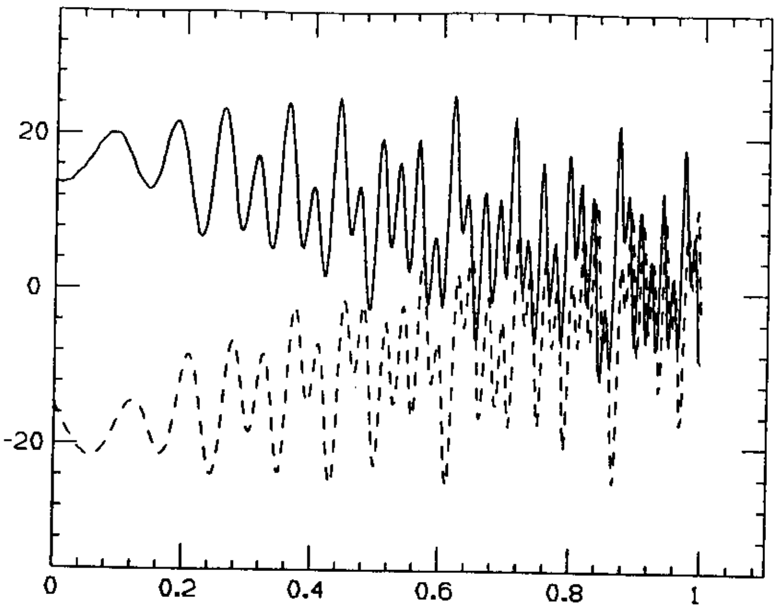

Figure 3 Re(aF) (dash) and Im(aF) (sld) vs. (a-r)/a

Equation (45) and $div\vec{E} = 0$ define E_z. This gives the impedance per period of the array:

$$Z_l(\omega) = \frac{2}{c}\int_0^\infty \frac{dq}{q} J_0^2(qa)(e^{-iLq^2/2k} - 1)$$

$$-\frac{2}{c}\int_0^\infty \frac{dq}{q} J_0(qa)(e^{-iLq^2/2k} - 1)[(1-\Lambda)J_0(qa) - \Lambda\int_0^a dr'\sqrt{\frac{r'}{a}}J_0(qr')F(a-r')]. \quad (55)$$

The first line is the impedance of a single cavity. The rest is the contribution of the field $f(k,r)$ on an iris. The first term in the brackets cancels the single cavity contribution. The estimate of the term with $F(r)$ shows that the last term in brackets is small. The remaining term has the same structure as that for a single cavity, but has the additional factor $\Lambda \propto 1/k$. Hence, the impedance of the periodic array decreases with frequencies as $k^{-3/2}$. For the real part we get

$$\mathrm{Re}Z(\omega) = Z_0\frac{2}{\sqrt{\pi}}(\frac{L}{ka^2})^{3/2} \quad (56)$$

whereas it goes down as $k^{-1/2}$ for a single cavity. The $k^{-3/2}$ behavior is the main result of the optical resonator model.[5]

The difference in the ω dependence of the impedances obtained with the optical resonator model and for a single cavity has been explained in our paper.[17] The impedance of the array with the arbitrary number M of periods is given in a closed form by an approximate solution of the exact system of equations obtained by field matching. It is shown that for $M \ll ka^2/L$ the impedance per period is the same as that for a single cavity. The length $ML \simeq ka^2$ is a typical coherent length of waves radiated with the Fraunhofer angle k/a along the axis. The same expression for $M \gg (ka^2/L)^{3/2}$ gives the impedance of the optical resonator model. The transition from the regime of a single cavity to the regime of a periodic array takes place for M in the interval $ka^2/L \ll M \ll (ka^2/L)^{3/2}$. For an accelerator with fixed length $ML > ka^2$ the impedance behaves asymptotically as $k^{-3/2}$.

Tapering–Comparison with the Diffractional Models

If a jump of a beam pipe radius is not abrupt, but tapered, the losses become smaller. There are no analytic results available on the effect of tapering today. Simulation with TBCI shows[14] that long smooth tapering can reduce the energy loss several times if bunches are not too short. From the results of the previous discussion we can expect that a taper affects the waves with small k for which the Fresnel diffraction angle $\theta \simeq 1/\sqrt{ka}$ is comparable or larger then the tapering angle.

Here we give a simple model which confirms this statement. This also gives us the opportunity to discuss the diffractional model of high-frequency impedances.[6,7] The radiated field satisfies the homogeneous wave equation. Hence, the radiated field in a volume is defined by its value on the surface around the volume:[18]

$$\vec{E}_\omega = \int dS'[ikG(\vec{n}' \times \vec{B}_\omega) + (\vec{n}'\vec{E}_\omega)\nabla' G + \vec{E}_\omega(\vec{n}'\vec{\nabla}')G - \vec{n}'(\vec{E}_\omega \vec{\nabla}' G)] \tag{57}$$

where $\vec{n}$ is the unit vector normal to the surface pointed inside of the volume and

$$G(\vec{r}, \vec{r}') = \frac{e^{ikR}}{4\pi R}, \quad R = |\vec{r} - \vec{r}'| = \sqrt{(z-z')^2 + r^2 + r'^{\,2} - 2rr'\cos\phi}$$

is the Green's function of the wave equation

$$(\Delta + k^2)G = -\delta(\vec{r} - \vec{r}').$$

It can be represented as well in the form

$$G = \frac{i}{8\pi}\int_{-\infty}^{\infty} dq e^{iq(z-z')}[J_0(rs)H_0^{(1)}(r's) + 2\sum_{m=1}^{\infty} J_m(rs)H_m^{(1)}(r's)\cos m\phi]\theta(r'-r) + (r \leftrightarrow r') \tag{58}$$

where $s = \sqrt{k^2 - q^2}$ and the path of integration goes below the cut for $r > k$, and above the cut for $r < k$. k^2 is understood here as having an infinitely small positive imaginary part.

Let us consider a tapered cavity. The radius $r(z)$ is changed as follows:

$$r = a_0 \quad z < 0; \quad r = a_0 + z\tan\alpha \quad 0 < z < g; \quad r = a_0 \quad z > g.$$

Maximum cavity radius is $b_0 = a_0 + g\tan\alpha$. Before we specify the fields on the surface of the tapered cavity let us consider the situation in the straight pipe. The electric field of a particle in the ultrarelativistic case has only radial (i.e., the normal to the beam pipe wall) components, which induces an image current in the wall. However, as it was noted before, the fields of the image current are small for $\gamma \to \infty$. The normal component of the induced electric field and the azimuthal component of the induced magnetic field remain zero. We assume that it is also true on the tapered surface. At the same time, the sum of the tangential component of the synchronous field of a particle

$$E_t^{\mathrm{part}} = (2e/cr)e^{ikz}\sin\alpha$$

and the tangential component of the field of the image current E_t on the metallic surface has to be zero. Therefore, $E_t = -E_t^{\mathrm{part}}$ and the fields on the tapered surface are

$$E_r = -E_t^{\mathrm{part}}\sin\alpha, \quad E_z = -E_t^{\mathrm{part}}\cos\alpha, \quad B_\phi = 0 \tag{59}$$

$E_t = 0$ on the beam pipe opening. Therefore, in particular, as previously considered we could neglect the reflected wave in the beam pipe at zero approximation.

Boundary conditions Eq. (59) are the same as were assumed in consideration of the diffraction on the washer previously in the section Impedances of a Cavity and a Step.

Equations (57), (58) and Eq. (59) give E_z component in the cavity as the surface integral over the metallic walls of the cavity

$$E_{\omega z} = \int dS \frac{\partial G}{\partial r'} E_t^{part} \tag{60}$$

or

$$E_{\omega z} = -\frac{ie}{4c}e^{ikz}\int_{a_0}^{b_0} dr' \int_{-\infty}^{\infty} dq e^{i(q-k)(z-z(r'))}\sqrt{k^2-q^2}J_0(r\sqrt{k^2-q^2})H_1^{(1)}(r'\sqrt{k^2-q^2}) \tag{61}$$

where $z(r') = (r'-a_0)\cot\alpha$. Integrating that over z, $0 < z < g$, with the offset $r = a_0$ we obtain the impedance

$$Z_l(k) = \frac{1}{4c}\int_{a_0}^{b_0} dr' \int_{-\infty}^{\infty} d\tau \frac{1-e^{ig(\tau-k)}}{k-\tau} e^{-i(\tau-k)z(r')}\sqrt{k^2-\tau^2}J_0(a_0\sqrt{k^2-\tau^2})H_1^{(1)}(r'\sqrt{k^2-\tau^2}). \tag{62}$$

The main contribution is given by $\tau \simeq \sqrt{k/g} \ll k$,

$$Z_l(k) \simeq \frac{-i}{2c}\int_0^\infty \frac{dq}{q}(1-e^{-igq^2/2k})J_0(qa_0)H_0^{(1)}(qa_0)\int_{a_0}^{b_0} q dr \sqrt{\frac{a_0}{r}} e^{iq(r-a_0)(1+\frac{q\cot\alpha}{2k})}.$$

For abrupt change of the radius $\alpha = \pi/2$ and that gives the same as Eq. (27) except that $J_0(y) - J_0(\frac{b}{a}y)$ in Eq. (27) is replaced by $H_0^{(1)}(y) - H_0^{(1)}(\frac{b}{a}x)$. This difference gives negligible small contribution to the impedance Eq. (29). Otherwise, Eq. (29) has to be multiplied by the factor

$$1 - \frac{1}{2\tan(\alpha)\sqrt{kg}}$$

Hence, tapering does not reduce the impedance if $\sqrt{kg}\tan\alpha \gg 1$. The same conclusion has been done in[21].

CONCLUSION

The main results of this paper are two-fold. First, the handy formulas describing the high frequency impedances of the typical elements have been rederived. We also discussed more difficult problems for which answers are unknown or not well established, such as the cross-talk between different elements of the system, transition to a periodic array, and tapering. Secondly, we formulated the iterative method of calculating high-frequency impedance for arbitrary cavity. In zero approximation it is given by Eq. (4) and Eq. (60). The field given by Eq. (60) is defined by matching the field E_z in the beam pipe. Maxwell equation $div\vec{E} = 0$ gives E_r in the beam pipe. After, this boundary condition can be found for the next iteration. The series obtained in this way are similar to the Born series of the theory of scattering. This approach may help choose the appropriate approximation in a more rigorous consideration of the problem.

There are some general recommendations for reduction of the impedances. The vacuum chamber must be as smooth as possible. All variations of the radius or cross-section of the beam-pipe have to be minimized. Cross-talk between cavities tends to reduce the total impedance. Supporting a cylindrycal symmetry is very important for reduction of the transverse impedance. The use of numeric codes is necessary for an accurate estimate of the impedances, especially elements without cylindrycal symmetry.

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REFERENCES

1. K. L. Bane, P. B. Wilson, T. Weiland, SLAC-PUB-3528, 1984
2. K. Bane, SLAC-PUB-4169, 1986
3. W. K. H. Panofsky and W. A. Wenzel, Rev. Sci. Instrum. 27, 967, (1956)
4. T. Weiland, Nucl. Instr. and Methods, 216 (1983), 31-34
5. L. A. Vainstein, Soviet Phys. JETP 17, 19863,
   E. Keil, Nucl. Instrum. Methods 100, 419, (1972)
6. K. Bane and M. Sands, SLAC-PUB-4441, 1987
7. R. B. Palmer SLAC-PUB-4433, 1987
8. V. E. Balakin, A. V. Novokhatsky, Proc. of the 12th Int. Conf. on High Energy Acceler., Fermilab, 1983
9. J. D. Lawson, RHEL/M 144, 1963
10. G. D. Dome, IEEE Trans. Nucl. Sci., Vol. NS-32, No. 5 (1985), p. 2531
11. S. A. Heifets and S. A. Kheifets, CEBAF-PR-87-030, 1987
12. K. Bane, a talk at the Workshop "Impedances Beyond Cutoff", Berkeley, 1987
    H. Henke, *ibid*
13. S. A. Kheifets and S. A. Heifets, SLAC-PUB-3965, 1986
14. J. J. Bisognano, S. A. Heifets and B. C. Yunn, CEBAF-PR-88, 1988
15. P. B. Wilson, LEP-70/62, 1978
16. L. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, 1980
17. S. A. Heifets and S. A. Kheifets, SLAC-PUB-4625, 1988
18. J. D. Jackson, Classical Electrodynamics, Second Edition
19. S. Kheifets, L. Palumbo, V. G. Vaccaro IEEE Trans. on Nuclear Sci., Vol. NS-34, No. 5, 1987
20. R. Gluckstern and B. Zotter LEP-Note 613, September 1988
21. R. L. Gluckstern High Frequency Behavior of the Longitudinal Impedance for a Cavity of General Shape, preprint, 1988